THE FAILURE OF THE SINGULAR CARDINAL HYPOTHESIS AND SCALES

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Abstract. Starting from a supercompact cardinal κ, we build a model, in which κ is singular string limit, the singular cardinal hypothesis fails at κ and there are no very good scales at κ. Moreover there is a bad scale at κ, and so weak square fails.

1. Introduction

The Singular Cardinal Problem is the problem to find a complete set of rules for the behavior of the operation $\kappa \mapsto 2^\kappa$ for singular cardinals κ. One central theme is how much "reflection-type" properties are consistent with the failure of the *singular cardinal hypothesis* (SCH). SCH states that if κ is singular and $2^{\mathrm{cf}(\kappa)} < \kappa$, then $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$. In particular, if κ is strong limit singular, then $2^\kappa = \kappa^+$.

Scales are a central concept in PCF theory. Given a singular cardinal $\kappa = \sup_n \kappa_n$, where each κ_n is regular, a *scale of length* κ^+ is a sequence of functions $\langle f_\alpha \mid \alpha < \kappa^+\rangle$ in $\prod_n \kappa_n$ that is increasing and cofinal with respect to the eventual domination ordering. A point $\alpha < \kappa^+$ with $\mathrm{cf}(\alpha) > \omega$ is *good* if there is an unbounded $A \subset \alpha$ such that $\{f_\beta(n) \mid \beta \in A\}$ is strictly increasing for all large n. If A is a club in α, then α is *very good*. A scale is *good*, resp. *very good*, if on a club every point of uncountable cofinality is good, resp. very good. A scale is *bad* if it is not good.

Very good scales follow from intermediate square principles, and in turn imply failure of simultaneous stationary reflection (Cummings-Foreman-Magidor, [2]). Thus the non existence of a very good scale is a "reflection-type" property, and it has been open whether it is consistent with the failure of SCH at a singular strong limit cardinal.

Extender based forcing, developed by Gitik and Magidor [5], violates SCH at a singular cardinal κ while keeping GCH below κ. The set up is to start with a singular κ, such that $\kappa = \sup_n \kappa_n$, each κ_n is a strong cardinal, and then force to add many sequences though κ, but without adding bounded subsets at κ. In his Ph.D. thesis [7], Assaf Sharon modified this forcing to construct a model, where SCH fails at κ and there are no very good scales at κ. In his model, however, bounded subsets of κ are added, and κ is no longer strong limit. More precisely, only κ_0 remains (regular) strong limit.

Another way to violate SCH is via Magidor's supercompact Prikry forcings, [6]. An important variation is Gitik-Sharon's diagonal supercompact Prikry, [4]. In [8], we defined a forcing notion, called *hybrid Prikry*, which combines the diagonal supercompact forcing from Gitik-Sharon [4] and the original extender based forcing. This poset simultaneously singularizes all cardinals in the interval $[\kappa, \kappa^{+\omega})$, for a large cardinal κ, and uses extenders to add many Prikry sequences to $\prod_n \kappa$, so that SCH is violated. Here we define a *modified hybrid Prikry forcing*, combining ideas from [8] and the modified extender based forcing

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from Assaf Sharon's thesis, [7]. We use it to obtain the consistency of not SCH and no very good scale at a singular strong limit.

Our main forcing does not add bounded subsets of κ, thereby keeping it strong limit. However, before the main forcing, we need some Laver type preparation, to achieve no very good scales. Thus we can't quite keep GCH below κ, but we do have GCH at every inaccessible $\alpha < \kappa$.

Theorem 1.1. *Starting from a supercompact, it is consistent that SCH fails at a strong limit κ, and there is no very good scale at κ.*

In our forcing extension, there is also a bad scale:

Theorem 1.2. *Starting from a supercompact, it is consistent that SCH fails at a strong limit κ, there is a bad scale at κ and for all inaccessible cardinals $\alpha < \kappa$, $2^\alpha = \alpha^+$.*

The existence of a bad scale implies that weak square fails in our model. The failure of SCH together with a bad scale was already achieved in [4] at $\aleph_{\omega^2}$. However, there is no natural way to modify their forcing for $\aleph_\omega$. Our construction gives a different strategy, leaving the possibility of pushing it down to $\aleph_\omega$ open.

A key difference between the original extender based forcing and Assaf Sharon's version is that in the latter case forcing with the direct extension order preserves κ^+. Similarly, starting from a supercompact κ, we will define a Prikry type forcing $(\mathbb{P}, \leq, \leq^*)$ with the key feature that both $(\mathbb{P}, \leq)$ and $(\mathbb{P}, \leq^*)$ preserve μ, where $\mu := (\kappa^+)^{V^{\mathbb{P}}}$.

In section 2 we define the forcing. Preservation of μ is shown in section 3, and the Prikry lemma is given in section 4. In section 5, we show that SCH fails in the generic extension. Section 6 has the proof that there is no very good scale in the generic extension, and in section 7 we define a bad scale.

2. The forcing

Suppose that GCH holds; let κ be supercompact, and let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of strong cardinals above κ. Denote $\kappa_\omega = \sup_n \kappa_n$, $\mu := \kappa_\omega^+$. For each $n < \omega$, let U_n be a normal measure on $\mathcal{P}_\kappa(\kappa_n)$, and let $j_n := j_{U_n}$.

Suppose also that $Ult_{U_n} \models \kappa_n$ is $j_n(\mu^+)$-strong. So for a U_n- measure one set of x's in $\mathcal{P}_\kappa(\kappa_n)$, $\kappa^n_x := o.t.(x)$ is a μ^+ - strong cardinal. Say this is witnessed by $j_x : V \to M_x$.

Let $x \in \mathcal{P}_\kappa(\kappa_n)$ be as above. Let $\langle E_{x\alpha} \mid \alpha < \mu^+ \rangle$ be κ^n_x complete ultrafilters on κ^n_x, where $E_{x\alpha} = \{Z \subset \kappa^n_x \mid \alpha \in j_x(Z)\}$. Arguing as in [3] we define a strengthening of the Rudin-Keisler order: for $\alpha, \beta < \mu^+$, set $\alpha \leq_{E_x} \beta$ if $\alpha \leq \beta$ and there is a function $f : \kappa^n_x \to \kappa^n_x$, such that $j_x(f)(\beta) = \alpha$. For $\alpha \leq_{E_x} \beta$, fix projections $\pi_{\beta\alpha} : \kappa^n_x \to \kappa^n_x$ to witness this ordering, setting $\pi_{\alpha,\alpha}$ to be the identity. We do this as in Section 2 of [3] with respect to κ^n_x, so that we have:

1. $j_x \pi_{\beta\alpha}(\beta) = \alpha$.
2. For all $a \subset \mu^+$ with $|a| < \kappa^n_x$, there are unboundedly many $\beta < \mu^+$, such that $\alpha <_{E_x} \beta$ for all $\alpha \in a$.
3. For $\alpha < \beta \leq \gamma$, if $\alpha \leq_{E_x} \gamma$ and $\beta \leq_{E_x} \gamma$, then $\{\nu < \kappa^n_x \mid \pi_{\gamma,\alpha}(\nu) < \pi_{\gamma,\beta}(\nu)\} \in E_{x\gamma}$.
4. If $\{\alpha_i \mid i < \tau\} \subset \alpha < \mu^+$ with $\tau < \kappa$, are such that for all $i < \tau$, $\alpha_i <_{E_x} \alpha$, then there is $A \in E_{x\alpha}$, such that for all $\nu \in A$, for all $i, j < \tau$, if $\alpha_i \leq_{E_x} \alpha_j$, then $\pi_{\alpha,\alpha_i}(\nu) = \pi_{\alpha_j,\alpha_i}(\pi_{\alpha,\alpha_j}(\nu))$.

2.1. **The modules.**

Definition 2.1. *The poset* $\mathbb{Q}^n_x = \mathbb{Q}^n_{x0} \cup \mathbb{Q}_{x1}$ *is defined as follows:*
$\mathbb{Q}^n_{x1} = \{f : \mu^+ \rightharpoonup \kappa^n_x \mid |f| \leq \kappa^n_x\}$ *and* $\leq_{x1}$ *is the usual ordering.* $\mathbb{Q}^n_{x0}$ *has conditions of the form* $p = \langle a, A, f\rangle$ *such that:*

- $a \subset \mu^+$, $|a| < \kappa^n_x$, *for all* $\beta \in a$, $\beta \leq_{E_x} \max(a)$,
- $f \in \mathbb{Q}^n_{x1}$, *and* $a \cap \mathrm{dom}(f) = \emptyset$
- $A \in E_{x \max a}$,
- *for all* $\alpha \leq \beta \leq_{E_x} \gamma$ *in* a, *for all* $\nu \in \pi_{\max a,\gamma}$"$A$, $\pi_{\gamma,\alpha}(\nu) = \pi_{\beta,\alpha}(\pi_{\gamma,\beta}(\nu))$.
- *for all* $\alpha < \beta$ *in* a, *for all* $\nu \in A$, $\pi_{\max a,\alpha}(\nu) < \pi_{\max a,\beta}(\nu)$

$\langle b, B, g\rangle \leq_{x0} \langle a, A, f\rangle$ *if:*

(1) $b \supset a$,
(2) $\pi_{\max b \max a}$"$B \subset A$,
(3) $g \supset f$.

Define $\leq^*_x = \leq_{x0} \cup \leq_{x1}$ *and for* $p, q \in \mathbb{Q}^n_x$, $p \leq_x q$, *if* $p \leq^*_x q$ *or* $p \in \mathbb{Q}^n_{x1}$, $q = \langle a, A, f\rangle \in \mathbb{Q}^n_{x0}$ *and*

(1) $p \supset f$, $a \subset \mathrm{dom}(p)$
(2) $p(\max a) \in A$
(3) *for all* $\beta \in a$, $p(\beta) = \pi_{\max a,\beta}(p(\max a))$.

Definition 2.2. *For a condition* $p = \langle a, A, f\rangle \in \mathbb{Q}^n_{x0}$ *and* $\nu \in A$, *let* $p^\frown\nu = f \cup \{\langle \beta, \pi_{\max a,\beta}(\nu)\rangle \mid \beta \in a\}$. *I.e.* $p^\frown\nu$ *is the weakest extension of* p *in* $\mathbb{Q}^n_{x1}$ *with* ν *in its range.*

Note that if $g \in \mathbb{Q}^n_{x1}$, with $g \leq p$, there is a unique $\nu \in A$ such that $g \leq p^\frown\nu$ (take $\nu = g(\max a)$).

Finally, for $n < \omega$, denote

$$\mathbb{Q}^n_0 := [x \mapsto \mathbb{Q}^n_{x0}]_{U_n}, \mathbb{Q}^n_1 := [x \mapsto \mathbb{Q}^n_{x1}]_{U_n}, \text{ and } \mathbb{Q}^n := [x \mapsto \mathbb{Q}^n_x]_{U_n}.$$

Since each $\mathbb{Q}^n_{x0}$ is κ^n_x-closed, we have that $\mathbb{Q}^n_0$ is κ_n-closed. Also, for each $\alpha = [x \mapsto \alpha_x]_{U_n} < j_n(\mu^+)$, set $E_{n\alpha} = [x \mapsto E_{x\alpha_x}]_{U_n}$. $\mathbb{Q}^n$ is the extender based module over κ_n with respect to $\langle E_{n\alpha} \mid \alpha < j_n(\mu^+)\rangle$ and Cohen parts of size less than or equal to κ_n.

2.2. **The main forcing.** For $x, y \in \mathcal{P}_\kappa(\kappa_\omega)$, we will denote $\kappa_x = \kappa \cap x$ and use the notation $x \prec y$ to mean $x \subset y$ and $o.t.(x) < \kappa_y$. Since on a measure one set, κ_x is an inaccessible cardinal, we assume this is always the case. Similarly, for each k, on a measure one set, for $x \in \mathcal{P}_\kappa(\kappa_k)$, $\kappa^k_x = o.t.(x)$ is strong. So we assume this is always the case, too.

Definition 2.3. *Suppose we have sets* $A_i \in U_i$, $B_i \in [x \mapsto E_{x,\alpha_x}]_{U_i}$ *where each* $\alpha_x \in \mu^+$. *Let* $[\prod_{i \geq l} A_i \times B_i]^{<\omega}$ *denote the set of all finite sequences* $\langle \vec{x}, \vec{\nu}\rangle$, *where for some* n,

(1) $\vec{x} = \langle x_l, ..., x_n\rangle$, *is such that each* $x_i \in A_i$ *and* $x_l \prec x_{l+1} \prec ... \prec x_n$,
(2) $\vec{\nu} = \langle \nu_l, ..., \nu_n\rangle$, *is increasing and such that each* $\nu_i \in B_i$,
(3) *each* $\nu_i \in x_i$.

For every i, $B_i \in [x \mapsto E_{x,\alpha_x}]_{U_i}$, and $\nu \in B_i$ as above, fix representative functions $x \mapsto \nu_x$, such that $\nu := [x \mapsto \nu_x]_{U_i}$.

Definition 2.4. *Conditions in $\mathbb{P}$ are of the form*

$$p = \langle x_0, f_0, ..., x_{l-1}, f_{l-1}, A_l, F_l, A_{l+1}, F_{l+1}, ...\rangle$$

where $l = \mathrm{lh}(p)$ and:

1. *For $n < l$, $x_n \in \mathcal{P}_\kappa(\kappa_n)$, and for $i < n$, $x_i \prec x_n$,*
2. *For $n \geq l$, $A_n \in U_n$, and $x_{l-1} \prec y$ for all $y \in A_l$.*
3. *For $n \geq l$, $\mathrm{dom}(F_n) = A_n$, and for $y \in A_n$, $F_n(y) = \langle a_n(y), A_n(y), f_n(y)\rangle$, where $a_n(y) \in [\mu^+]^{<\kappa_y^n}$, $A_n(y) \in E_{y,\max(a_n(y))}$.*
   Denote $B_n := [y \mapsto A_n(y)]_{U_n}$.
4. *For $n < l$, $\mathrm{dom}(f_n) = \mathcal{P}_\kappa(\kappa_n)$, for every $x \in \mathcal{P}_\kappa(\kappa_n)$,*
   $$f_n(x) : [\prod_{i \geq l} A_i \times B_i]^{<\omega} \cap \{\langle \vec{z}, \vec{\nu}\rangle \mid x \prec \vec{z}\} \to \mathbb{Q}^n_{x1},$$
   and for $\langle \vec{z}, \vec{\nu}\rangle \subset \langle \vec{z'}, \vec{\nu'}\rangle$, $f_n(x)(\langle \vec{z'}, \vec{\nu'}\rangle) \leq f_n(x)(\langle \vec{z}, \vec{\nu}\rangle)$.
5. *For $n \geq l$, $F_n(y) = \langle a_n(y), A_n(y), f_n(y)\rangle$, such that:*
   (a) $\mathrm{dom}(f_n(y)) = [\prod_{i>n} A_i \times B_i]^{<\omega} \cap \{\langle \vec{z}, \vec{\nu}\rangle \mid y \prec \vec{z}\}$ *and for each* $\langle \vec{z}, \vec{\nu}\rangle \in \mathrm{dom}(f_n(y))$,
   $$\langle a_n(y), A_n(y), f_n(y)(\langle \vec{z}, \vec{\nu}\rangle)\rangle \in \mathbb{Q}^n_{y0},$$
   (b) *if $\langle \vec{z}, \vec{\nu}\rangle \subset \langle \vec{z'}, \vec{\nu'}\rangle$, $f_n(y)(\langle \vec{z'}, \vec{\nu'}\rangle) \leq f_n(y)(\langle \vec{z}, \vec{\nu}\rangle)$.*
6. *For $l \leq n < m, y \in A_n, y' \in A_m, y \prec y'$, we have $a_n(y) \subset a_m(y')$*

For a condition p as above we will use $f^p_n, x^p_n, n < \mathrm{lh}(p)$ and $A^p_n, B^p_n, F^p_n, F^p_n(y) = \langle a^p_n(y), A^p_n(y), f^p_n(y)\rangle, n \geq \mathrm{lh}(p)$ to denote its components as defined above. Also for $n \geq \mathrm{lh}(p)$, let $\beta^p_n := [x \mapsto \max(a^p_n(x))]_{U_n}$.

We say that $q \leq^* p$ if $\mathrm{lh}(q) = \mathrm{lh}(p) = l$, and

1. for all $n < l$, $x^q_n = x^p_n$ and for $n \geq l$, $A^q_n \subset A^p_n$,
2. for all $n \geq l$, $y \in A^q_n$, $a^q_n(y) \supset a^p_n(y), \pi_{\max(a^q_n(y), a^p_n(y))}{}''A^q_n(y) \subset A^p_n(y)$.

   For $n \geq l$ and $\vec{\nu} \in \prod_{n \leq i \leq k} B^p_i$, denote $\pi(\vec{\nu}) = \langle \pi_{\beta^q_n, \beta^p_n}(\nu_n), ..., \pi_{\beta^q_k, \beta^p_k}(\nu_k)\rangle$.
3. for all $n < l$, $\forall_{U_n} x$, for all $\langle \vec{z}, \vec{\nu}\rangle \in \mathrm{dom}(f^q_n(x))$, $f^q_n(x)(\langle \vec{z}, \vec{\nu}\rangle) \leq_{\mathbb{Q}^n_{x1}} f^p_n(x)(\langle \vec{z}, \pi(\vec{\nu})\rangle)$,
4. for all $n \geq l$, $y \in A^q_n$ and $\langle \vec{z}, \vec{\nu}\rangle \in \mathrm{dom}(f^q_n(y))$,
   $$\langle a^q_n(y), A^q_n(y), f^q_n(y)(\langle \vec{z}, \vec{\nu}\rangle)\rangle \leq_{\mathbb{Q}^n_{y0}} \langle a^p_n(y), A^p_n(y), f^p_n(y)(\langle \vec{z}, \pi(\vec{\nu})\rangle)\rangle.$$
5. for all $n \geq l$, $A^q_n \subset \{x \mid \nu \in x \to \pi_{\beta^q_n, \beta^p_n}(\nu) \in x\}$. This is needed to ensure transitivity of $\leq_{\mathbb{P}}$.

Suppose p has length l, $k > l$, and $\langle \vec{x}, \vec{\nu}\rangle \in [\prod_{i \geq l} A^p_i \times B^p_i]^{<\omega}$; $\vec{x} := \langle x_l, ..., x_{k-1}\rangle$, $\vec{\nu} := \langle \nu_l, ..., \nu_{k-1}\rangle$. We define the weakest $k-l$-step extension of p obtained from $\langle \vec{x}, \vec{\nu}\rangle$ denoted by $p^\frown\langle \vec{x}, \vec{\nu}\rangle$ to be the condition

$$\langle x^p_0, f_0, ..., x^p_{l-1}, f_{l-1}, x_l, f_l, ..., x_{k-1}, f_{k-1}, A_k, F_k, A_{k+1}, F_{k+1}, ...\rangle,$$

such that:

1. for $n \geq k$, $A_n = A^p_n \cap \{y \mid x_{k-1} \prec y\}$,
2. for $n < l$, for $x \in \mathcal{P}_\kappa(\kappa_n)$, and $\langle \vec{z}, \vec{\delta}\rangle \in \mathrm{dom}(f_n(x))$, $f_n(x)(\langle \vec{z}, \vec{\delta}\rangle) = f^p_n(x)(\langle \vec{x}, \vec{\nu}\rangle^\frown\langle \vec{z}, \vec{\delta}\rangle)$,

(3) for $l \leq n < k$, for $x \in \mathcal{P}_\kappa(\kappa_n)$ with $\nu_x \in A^p_n(x)$, for all $\langle \vec{z}, \vec{\delta} \rangle \in \mathrm{dom}(f_n(x))$,
$$f_n(x)(\langle \vec{z}, \vec{\delta} \rangle) = \langle a^p_n(x), A^p_n(x), f^p_n(x)(\langle x_{n+1}, ..., x_{k-1}, \nu_{n+1}, ...\nu_{k-1} \rangle^\frown \langle \vec{z}, \vec{\delta} \rangle)\rangle^\frown \nu_x;$$
otherwise, if $\nu_x \notin A^p_n(x)$, for all $\langle \vec{z}, \vec{\delta} \rangle \in \mathrm{dom}(f_n(x))$, set $f_n(x)(\langle \vec{z}, \vec{\delta} \rangle) = \emptyset$.

(4) for $n \geq k$ and $y \in A_n$, we have $F_n(y) = F^p_n(y)$.

We can finally define the full ordering:

Definition 2.5. *$q \leq p$ if $q \leq^* p$ or for some $\langle \vec{y}, \vec{\nu} \rangle$, we have that $q \leq^* p^\frown \langle \vec{y}, \vec{\nu} \rangle$.*

3. Preservation of μ

Define $\mathbb{P}_n := \{p \in \mathbb{P} \mid \mathrm{lh}(p) = n\}$. We will show that $(\mathbb{P}_0, \leq^*)$ preserves μ. The idea will be that for every n, we can regard it as a combination of two subposets, one with κ_n^{++}-c.c., and the other κ_{n+1}-closed. We use this to show that $(\mathbb{P}_0, \leq^*)$ preserves κ_n for every n, and then conclude that it must preserve μ. We remark that our arguments can be adapted to show $(\mathbb{P}_k, \leq^*)$ preserves μ, for every $k < \omega$.

Definition 3.1. *For p, q in $\mathbb{P}_0$, we say that $p \sim q$ if for all $k < \omega$, $B^p_k = B^q_k = B_k$, and there are measure one sets $A_k \subset A^p_k \cap A^q_k$, such that for all $k < \omega, x \in A_k, \langle \vec{z}, \vec{\nu} \rangle \in [\prod_{i>k} A_i \times B_i]^{<\omega}$ with $x \prec \vec{z}$, we have that $a^p_k(x) = a^q_k(x), A^p_k(x) = A^q_k(x), f^p_k(x)(\langle \vec{z}, \vec{\nu} \rangle) = f^q_k(x)(\langle \vec{z}, \vec{\nu} \rangle)$. Define $p \leq^\sim q$ if there is $p' \sim p$ with $p' \leq q$.*

Let $\mathbb{P}_0 \restriction [n, \omega) := \{\langle A^p_n, F^p_n, A^p_{n+1}, F^p_{n+1}, ...\rangle \mid p \in \mathbb{P}_0\}$ with the induced ordering from $\leq^\sim$ (which we denote the same). Note that $\langle \mathbb{P}_0, \leq^* \rangle$ and $\langle \mathbb{P}_0, \leq^\sim \rangle$ are isomorphic.

Proposition 3.2. *$\langle \mathbb{P}_0 \restriction [n, \omega), \leq^\sim \rangle$ is κ_n-closed, for all $n \geq 0$.*

Proof. Suppose that $\tau < \kappa_n$ and $\langle p_\eta \mid \eta < \tau \rangle$ is a $\leq^\sim$-decreasing sequence in $\mathbb{P}_0 \restriction [n, \omega)$. For each $\eta < \delta$, let $\langle A^{\eta,\delta}_k \mid n \leq k < \omega \rangle$ be measure one sets in U_k, respectively, witnessing that $p^\delta \leq^\sim p^\eta$.

For $k \geq n$, set $A^p_k = \triangle A^{\eta,\delta}_k = \{x \mid x \in \bigcap_{\eta \in x, \delta \in x} A^{\eta,\delta}_k\}$. Let $\bar{m} < \mu^+$ be above the supremum of all of the domains of the $f^{p_\eta}_k$'s, i.e.
$$\bar{m} > \sup_{\eta<\tau, k<\omega, x \in A^{p_\eta}_k, \langle \vec{z}, \vec{\nu} \rangle \in \mathrm{dom}(f^{p_\eta}_k(x))} \mathrm{dom}(f^{p_\eta}_k(x)(\langle \vec{z}, \vec{\nu} \rangle)).$$

Inductively on k, for all $x \in A^p_k$, set
$$a^p_k(x) = \bigcup_{\eta \in x \cap \tau} a^{p_\eta}_k(x) \cup \bigcup_{n \leq m < k, w \in A^p_m, w \prec x} a^p_m(w) \cup \{m\},$$
where m is a maximal element above $\bar{m}$. Then let $A^p_k(x) = \bigcap_{\eta \in x \cap \tau} \pi^{-1}_\eta (A^{p_\eta}_k(x))$, where $\pi_\eta = \pi_{\max(a^p_k(x)), \max(a^{p_\eta}_k(x))}$.

Now let $B'_i := [y \mapsto A^p_i(y)]_{U_i}$. For all $\langle \vec{z}, \vec{\nu} \rangle \in [\prod_{i>k} A^p_i \times B'_i]^{<\omega}$ with $x \prec \vec{z}$, define
$$f^p_k(x)(\langle \vec{z}, \vec{\nu} \rangle) = \bigcup_{\eta \in x \cap \tau} f^{p_\eta}_k(x)(\langle \vec{z}, \pi^\eta(\vec{\nu}) \rangle),$$
where π^η is the corresponding pointwise projections from the maximal coordinates of p to the maximal coordinates of p_η.

We claim that p is as desired. For if $\eta < \tau$, for $k \geq n$, let $A_k = A^p_k \cap \{x \mid \eta \in x\} \cap \{x \mid \nu \in x \rightarrow \pi_{\beta^p_k, \beta^{p_\eta}_k}(\nu) \in x\}$. Then $\langle A_k \mid k \geq n \rangle$ witness that $p \leq^\sim p_\eta$.

$\square$

For $n > 0$ and $p \in \mathbb{P}_0$, let $\pi_n(p) = \langle A_0^p, F_0^p, A_1^p, F_1^p, ...A_{n-1}^p, F_{n-1}^p \rangle$. Set $\mathbb{P}_{0n} := \{\pi_n(p) \mid p \in \mathbb{P}_0\}$ with the natural induced ordering from $\leq^*$.

Proposition 3.3. *For all $n \geq 0$, $\mathbb{P}_{0n+1}$ has the κ_n^{++} c.c.*

Proof. By induction on n. Suppose for contradiction that $\{\pi_{n+1}(p_\eta) \mid \eta < \kappa_n^{++}\}$ is an antichain in $\mathbb{P}_{0n+1}$. By strengthening each p_η if necessary, we may assume that the part above n is the same, i.e. for all $i > n$, $[F_i^{p_\eta}]_{U_i} = [F_i]_{U_i}$ for all η. For $i > n$, denote $[F_i]_{U_i} := \langle a_i^*, B_i, f_i^* \rangle$, and let $\alpha_i = \max(a_i^*)$. Then each $B_i \in E_{i,\alpha_i}$. For $m > n$, set $i_m := j_{E_{n+1,\alpha_{n+1}}} \circ j_{n+1} \circ ... j_{E_{m,\alpha_m}} \circ j_m$.

Fix $x \in A_n^{p_\eta}$. We will define functions $f_{x,m}^\eta$ for $n < m$ as follows.

- If $m = n+1$, for $\nu \in B_{n+1}$, let $f_{x,\nu,n+1}^\eta := [z \mapsto f_n^{p_\eta}(x)(\langle z, \nu \rangle)]_{U_{n+1}}$. $|f_n^{p_\eta}(x)(\langle z, \nu \rangle)| \leq \kappa_x^n$, so $|f_{x,\nu,n+1}^\eta| \leq \kappa_x^n$. Then let $f_{x,n+1}^\eta := [\nu \mapsto f_{x,\nu,n+1}^\eta]_{E_{n+1,\alpha_{n+1}}}$. Again, we have that $|f_{x,n+1}^\eta| \leq \kappa_x^n$
- If $m = n+2$, for $\nu \in B_{n+1}$, $y \in A_{n+1}^{p_\eta}$, $\delta \in B_{n+2}$, with $\nu \in y$, let,
  - $f_{x,\nu,y,\delta}^\eta := [z \mapsto f_n^{p_\eta}(x)(\langle y, z, \nu, \delta \rangle)]_{U_{n+2}}$;
  - $f_{x,\nu,y}^\eta := [\delta \mapsto f_{x,\nu,y,\delta}^\eta]_{E_{n+2,\alpha_{n+2}}}$;
  - $f_{x,\nu}^\eta := [y \mapsto f_{x,\nu,y}^\eta]_{U_{n+1}}$;
  - $f_{x,n+2}^\eta := [\nu \mapsto f_{x,\nu}^\eta]_{E_{n+1,\alpha_{n+1}}}$.

  As before, $|f_{x,n+2}^\eta| \leq \kappa_x^n$.
- ...

  Continue in a similar fashion for all $m > n$.

Then each $f_{x,m}^\eta$ is a partial function from $i_m(\mu^+)$ to κ_x^n of size less than or equal to κ_x^n. Define a partial function $F_m^\eta : \mathcal{P}_\kappa(\kappa_n) \times i_m(\mu^+) \rightharpoonup \{Y\} \cup \kappa$ by:

$$F_m^\eta(x, \alpha) := \begin{cases} Y & \text{if } \alpha \in i_m(a_n^{p_\eta}(x)) \\ f_{x,m}^\eta(\alpha) & \text{if } \alpha \in \mathrm{dom}(f_{x,m}^\eta) \end{cases}$$

Let F^η be the function given by $F^\eta(m, x, \alpha) = F_m^\eta(x, \alpha)$. This is a function of size less than κ_n^+. So, by applying the Δ-system lemma, we get an unbounded $I \subset \kappa_n^{++}$, such that $\langle F^\eta \mid \eta \in I \rangle$ forms a Δ system, and the functions have the same value on the kernel. Note that this implies that for all η, δ in I and for all $n < m, x \in \mathcal{P}_\kappa(\kappa_n)$, $i_m(a_n^{p_\eta}(x)) \cap \mathrm{dom}(f_{x,m}^\delta) = \emptyset$.

By the inductive hypothesis, if $n > 0$, $\mathbb{P}_{0n}$ has the κ_{n-1}^{++}-c.c. So let η, δ be distinct points in I, such that if $n > 0$, $\pi_n(p_\eta)$ and $\pi_n(p_\delta)$ are compatible. We will construct $p \in \mathbb{P}_0$, such that $\pi_{n+1}(p)$ is a common extension of of $\pi_{n+1}(p_\eta)$ and $\pi_{n+1}(p_\delta)$.

Let $\bar{m} < \mu^+$ be above the supremum of the domains of $f_k^{p_\eta}(x)(h)$ and $f_k^{p_\delta}(x)(h)$, for $k \leq n, x \in A_k^{p_\eta} \cap A_k^{p_\delta}, h \in \mathrm{dom}(f_k^{p_\eta}(x)) \cap \mathrm{dom}(f_k^{p_\delta}(x))$. Also, let r be a common extension of $\pi_n(p_\eta)$ and $\pi_n(p_\delta)$, such that for all $k < n$, $x \in A_k^r$, $a_k^r(x) = a_k^{p_\eta}(x) \cup a_k^{p_\delta}(x) \cup c$, where $c \subset \mu^+ \setminus \bar{m}$. We will define p so that $p \restriction n \sim r$.

For $i < n$, set $A_i^p = A_i^r$, for $x \in A_i^p$, set $a_i^p(x) = a_i^r(x), A_i^p(x) = A_i^r(x)$. And then $B_i^p = B_i^r$.

Also set $A_n^p = A_n^r \subset A_n^{p_\eta} \cap A_n^{p_\delta}$. For $x \in A_n^p$, let

$$a_n^p(x) = a_n^{p_\eta}(x) \cup a_n^{p_\delta}(x) \cup \bigcup_{i<n, w \in A_i^p, w \prec x} a_i^p(w) \cup \{m'\},$$

where $m' > \bar{m}$ is a maximal element in the extender ordering. Then, set

$$A^p_n(x) = \pi^{-1}_{m',m_\eta}(A^{p_\eta}_n(x)) \cap \pi^{-1}_{m',m_\delta}(A^{p_\delta}_n(x)),$$

where m_η, m_δ are the maximal elements of $a^{p_\eta}_n(x)$ and $a^{p_\delta}_n(x)$ respectively. Finally, for all $m > n$, let

$$f_{x,m} = f^\eta_{x,m} \cup f^\delta_{x,m}.$$

This is a well-defined function because the values on the kernel of the Δ system obtained above are the same.

Denote:

- $f_{x,m} = [\nu \mapsto f^m_n(x)(\nu)]_{E_{n+1,\alpha_{n+1}}}$;
- $f^m_n(x)(\nu) = [y \mapsto f^m_n(x)(\nu)(y)]_{U_{n+1}}$;
- $f^m_n(x)(\nu)(y) := [\delta \mapsto f^m_n(x)(\nu)(y)(\delta)]_{E_{n+2,\alpha_{n+2}}}$;
- $f^m_n(x)(\nu)(y)(\delta) := [z \mapsto f^m_n(x)(\nu)(y)(\delta)(z)]_{U_{n+2}}$;
- ... and so on until we reach m.

Then we have that:

$\forall^*_{E_{n+1,\alpha_{n+1}}} \nu_{n+1} \forall^*_{U_{n+1}} y_{n+1} \forall^*_{E_{n+2,\alpha_{n+2}}} \nu_{n+2} \forall^*_{U_{n+2}} y_{n+2} ... \forall^*_{E_m,\alpha_m} \nu_m \forall^*_{U_m} y_m$

$(\dagger) : f^m_n(x)(\nu_{n+1})(y_{n+1})...(\nu_m)(y_m) =$
$f^{p_\eta}_n(x)(\langle y_{n+1}, ..., y_m, \nu_{n+1}, ..., \nu_m\rangle) \cup f^{p_\delta}_n(x)(\langle y_{n+1}, ..., y_m, \nu_{n+1}, ..., \nu_m\rangle)$

and

$$\mathrm{dom}(f^m_n(x)(\nu_{n+1})(y_{n+1})...(\nu_m)(y_m)) \cap a^p_n(x) = \emptyset.$$

Then by taking diagonal intersection, for all $x \in A^{p_\eta}_k \cap A^{p_\delta}_k$, for all $m > n$, we have measure one sets $A^{x,m}_{n+1}, A^{x,m}_{n+2}, ..., A^{x,m}_m$ and $B^{x,m}_{n+1}, B^{x,m}_{n+2}, ..., B^{x,m}_m$, where each $A^{x,m}_i \in U_i$, $B^{x,m}_i \in E_{i,\alpha_i}$, such that for all $\langle \vec{y}, \vec{\nu}\rangle \in [\prod_{n<i\leq m} A^{x,m}_i \times B^{x,m}_i]^{<\omega}$ with $x \prec \vec{y}$, we have that the above equality holds.

We illustrate how these sets are defined for $m = n + 2$:

- $B_{n+1} = \{\nu \mid \forall^* y, \forall^* \delta, \forall^* z (\dagger) \text{ holds for } \langle y, z, \nu, \delta\rangle\}$.
- For every $\nu \in B_{n+1}$, let $A_\nu \in U_{n+1}$ witness it.
  Set $A_{n+1} = \triangle A_\nu = \{y \mid y \in \bigcap_{\nu \in y} A_\nu\} \in U_{n+1}$;
- For all $\nu \in B_{n+1}$, for all $y \in A_\nu$, let $B_{\nu,y} \in E_{n+2,\alpha_{n+2}}$ witness it.
  Set $B_{n+2} = \bigcap_{\nu,y} B_{\nu,y} \in E_{n+2,\alpha_{n+2}}$;
- For all $\nu \in B_{n+1}$, $y \in A_\nu$, and $\delta \in B_{\nu,y}$, let $A_{\nu,y,\delta} \in U_{n+2}$ witness it.
  Set $A_{n+2} := \triangle A_{\nu,y,\delta} = \{z \mid z \in \bigcap_{\delta \in z, y \prec z, \nu \in y} A_{\nu,y,\delta}\} \in U_{n+2}$;

For such x, for $i > n$, let

$$A^x_i = \bigcap_{i \leq m < \omega} A^{x,m}_i, B^x_i = \bigcap_{i \leq m < \omega} B^{x,m}_i.$$

Then set $A_i = \triangle A^x_i, B^p_i = \bigcap_{x \in \mathcal{P}_\kappa(\kappa_n)} B^x_i$.
For $n < i < \omega$, let $A^p_i = A_i \cap \{x \mid \nu \in x \to (\pi_{\beta^p_i, \beta^{p_\eta}_i}(\nu) \in x, \pi_{\beta^p_i, \beta^{p_\delta}_i}(\nu) \in x)\}$.
For $i \leq n$, let $F^p_i(y)$ be obtained from $F^r_i(y)$, restricted to B^p_i's.
For $x \in A^p_n$, $m > n$, and $\langle \vec{y}, \vec{\nu}\rangle$ in $[\prod_{i>n} A^p_i \times B^p_i]^{<\omega}$ with $x \prec \vec{y}$, let

$$f^p_n(x)(\langle y_{n+1}, ..., y_m, \nu_{n+1}, ..., \nu_m\rangle) = f^m_n(x)(\nu_{n+1})(y_{n+1})...(\nu_m)(y_m).$$

Then p is as desired.

$\square$

Using a similar, and actually simpler argument, we get:

Lemma 3.4. *Both* $(\mathbb{P}_0, \leq^*)$ *and* $(\mathbb{P}, \leq)$ *have the* μ^+*-c.c.*

Lemma 3.5. *Let* $n > 0$*.* $(\mathbb{P}_0, \leq^*)$ *preserves cardinals in the interval* $[\kappa_n^{++}, \kappa_{n+1}]$*.*

Proof. Suppose otherwise. Let n be such that some regular V-cardinal $\tau \in [\kappa_n^{++}, \kappa_{n+1}]$ is collapsed. Let $p \in \mathbb{P}_0$, and $\lambda < \tau$ be such that $p \Vdash_{\mathbb{P}_0} \dot{h} : \lambda \to \tau$ is onto. Fix $\alpha < \lambda$. We will define $\theta \leq \kappa_n^{++}$ and $\langle p_\eta, \alpha_\eta \mid \eta < \theta\rangle$ by induction of η, such that:

(1) $p_\eta \in \mathbb{P}_0, p_\eta \leq^* p$, $\alpha_\eta \in \tau$,
(2) $\langle p_\eta \restriction [n+1, \omega) \mid \eta < \theta\rangle$ is $\leq^\sim$-decreasing,
(3) $p_\eta \Vdash_{\mathbb{P}_0} \dot{h}(\alpha) = \alpha_\eta$.

Let α_0 and $p_0 \leq^* p$ be such that $p_0 \Vdash_{\mathbb{P}_0} \dot{h}(\alpha) = \alpha_0$. Suppose we have defined p_ξ, α_ξ, for all $\xi < \eta$. If $\eta = \kappa_n^{++}$, set $\theta = \eta$ and stop. Otherwise let $q \leq^* p$ be such that $q \restriction n+1 = p \restriction n+1$ and $q \restriction [n+1, \omega) \leq^\sim p_\xi \restriction [n+1, \omega)$ for all $\xi < \eta$. We can find such a condition because $\langle \mathbb{P}_0 \restriction [n+1, \omega), \leq^\sim\rangle$ is κ_{n+1}-closed.

Suppose that there is $r \in \mathbb{P}_0, r \leq^* q$ and $\beta \notin \{\alpha_\xi \mid \xi < \eta\}$, such that $r \Vdash \dot{h}(\alpha) = \beta$. Then let $\alpha_\eta = \beta$ and $p_\eta = r$. Otherwise, set $\theta = \eta$, $q_\alpha := q$, and stop.

Claim 3.6. $\theta < \kappa_n^{++}$*.*

Proof. Otherwise $\langle \pi_{n+1}(p_\eta) \mid \eta < \kappa_n^{++}\rangle$ is an antichain in $\mathbb{P}_{0n+1}$ of size κ_n^{++}. Contradiction with Proposition 3.3.

□

It follows that each q_α is defined. Note that for all α, $q_\alpha \restriction n+1 = p \restriction n+1$. Let $X_\alpha = \{\alpha_\eta \mid \eta < \theta\}$. Then $q_\alpha \Vdash \dot{h}(\alpha) \in X_\alpha$. Doing this inductively on $\alpha < \lambda$, we arrange that $\langle q_\alpha \restriction [n+1, \omega) \mid \alpha < \kappa\rangle$ is $\leq^\sim$-decreasing. Finally let $X = \bigcup_{\alpha<\lambda} X_\alpha$, and let $q \leq^* p$ be such that for all $\alpha < \lambda$, $q \restriction [n+1, \omega) \leq^\sim q_\alpha \restriction [n+1, \omega)$ and $q \restriction n+1 = p \restriction n+1$. Then $q \Vdash_{\mathbb{P}_0} \mathrm{ran}(\dot{h}) \subset X$, but $|X| < \tau$. Contradiction. □

Corollary 3.7. $\mathbb{P}_0$ *preserves* μ*.*

For conditions $p, q \in \mathbb{P}$, we say that p and q are *tail equivalent*, if for some large enough n, $p \restriction [n, \omega) \sim q \restriction [n, \omega)$, as defined earlier, restricted to $\mathbb{P} \restriction [n, \omega)$. In this case we write $p \sim_t q$. Denote the tail-equivalence class of p, by $t(p) := \{q \mid p \sim_t q\}$.

Definition 3.8. *Let* $\mathbb{D} := \{t(p) \mid p \in \mathbb{P}\}$ *with the ordering* $t(p) \leq_{\mathbb{D}} t(q)$ *if for some* n*,* $p \restriction [n, \omega) \leq^\sim q \restriction [n, \omega)$*.*

By considering the map $p \mapsto t(p)$, we get the following:

Proposition 3.9. *Both* $\mathbb{P}$ *and* $\mathbb{P}_0$ *project to* $\mathbb{D}$*.*

Proposition 3.10. *Suppose that* H *is* $\mathbb{D}$*-generic,* G_0 *is* $\mathbb{P}_0/H$*-generic, and* $p \in \mathbb{P}/H$*. Then there is some* n*, such that* $1^\frown p \restriction [n, \omega) \in G_0$*.*

Lemma 3.11. *Let* H *be* $\mathbb{D}$*-generic.* $\mathbb{P}/H$ *has the* μ*-c.c.*

Proof. Suppose $\{p_\eta \mid \eta < \mu\}$ are conditions in $\mathbb{P}/H$. I.e. for each η, $t(p_\eta) \in H$. By passing to an unbounded subset of μ, we may assume that there is $\bar{n} < \omega$, and $\vec{x}$ of length $\bar{n}$, such that all conditions have length $\bar{n}$ and Prikry stem $\vec{x}$. Let G_0 be $\mathbb{P}_0/H$-generic. Then for all η, there is some $n_\eta > \bar{n}$, such that $1^\frown p_\eta \restriction [n_\eta, \omega) \in G_0$.

Since in $V[G_0]$, μ is a regular cardinal, there is some unbounded $I \subset \mu$, such that for all $\eta \in I$, $n_\eta = n$. Now run a Δ-system argument for $\{p_\eta \upharpoonright n \mid \eta \in I\}$ in $V[G_0]$. This is similar (and actually simpler) to what was done in Proposition 3.3. Then we can find $\eta < \delta$, in I, such that $p_\eta \upharpoonright n, p_\delta \upharpoonright n$ have a common extension in $\mathbb{P} \upharpoonright n$. Let r be such an extension, and let $q \in \mathbb{P} \upharpoonright [n, \omega)$ be a common extension of $p_\eta \upharpoonright [n, \omega)$ and $p_\delta \upharpoonright [n, \omega)$. Then $r^\frown q$ is a common extension of p_η, p_δ.

□

Corollary 3.12. *$\mathbb{P}$ preserves μ.*

4. The Prikry lemma

First we show the diagonal lemma:

Lemma 4.1. *Suppose that p is a condition of length l and for all $\langle x, \nu\rangle \in A^p_l \times B^p_l$ with $\nu \in x$, we have $p_{x,\nu} \leq^* p^\frown\langle x, \nu\rangle$. Suppose also that:*

1. *There are $\langle \beta_n \mid l < n < \omega\rangle$, such that every $\beta_n^{p_{x,\nu}} \leq_{E_n} \beta_n$, and for all $y \in A^q_l$, for all h, with $y \prec h$,*
$$\langle f_l^{p_{x,\nu}}(y)(\pi^{x,\nu}(h)) \upharpoonright \mathrm{dom}(f_l^{p_{x,\nu}}(y)(\pi^{x,\nu}(h))) \setminus a^p_l(y) \mid \nu \in x, x \prec y\rangle$$
are pairwise compatible, where $\pi^{x,\nu}$ is the projection from the β_n's to the $\beta_n^{p_{x,\nu}}$'s.
2. *$\langle p_{x,\nu} \upharpoonright [l+1, \omega)\rangle$ are $\leq^\sim$ -pairwise compatible.*

Then there is a direct extension $q \leq^* p$, such that if r is a nondirect extension of q, then for some x, ν, we have that $r \leq p_{x,\nu}$. Moreover, we can choose q, so that for all $x \in A^q_l$, $a^q_l(x) = a^p_l(x)$.

Proof. For simplicity assume that $\mathrm{lh}(p) = 1$. Denote $p_{x,\nu} = \langle x_0, f_0^{x,\nu}, x, f_1^{x,\nu}, A_2^{x,\nu}, F_2^{x,\nu}, ...\rangle$, and for $n > 1$, $F_n^{x,\nu}(y) = \langle a_n^{x,\nu}(y), A_n^{x,\nu}(y), f_n^{x,\nu}(y)\rangle$. By taking diagonal intersections, by item (2), we can assume that for all $n > 1$, for all $\nu \in x, \delta \in w$, for all $y \in A^p_n$ with $x \prec y, z \prec y$ and for all h with $y \prec h$, $\langle a_n^{w,\delta}(y), A_n^{w,\delta}(y), f_n^{w,\delta}(y)(\pi_1(h))\rangle$ and $\langle a_n^{x,\nu}(y), A_n^{x,\nu}(y), f_n^{x,\nu}(y)(\pi_2(h))\rangle$ are pairwise compatible, where π_1 and π_2 project to the maximal coordinates of $p^{w,\delta}$ and $p^{x,\nu}$, respectively, from some coordinate above both.

For every ν, we have that $B_\nu := \{x \in A^p_1 \mid \nu_x \in A^p_1(x)\} \in U_1$. Set $A^q_1 = \triangle_\nu B_\nu$. For $y \in A^q_1$, set $a^q_1(y) = a^p_1(y), A^q_1(y) = A^p_1(y)$. For $n > 1$, let $A'_n = \triangle A_n^{x,\nu} := \{z \mid z \in \bigcap_{x \prec z, \nu \in x} A_n^{x,\nu}\}$. For $n > 1$ and $y \in A'_n$, set:

1. $a^q_n(y) \supset \bigcup_{x \prec y, \nu \in x} a_n^{x,\nu}(y)$, and
2. $A^q_n(y) = \bigcap_{x \prec y, \nu \in x} \pi^{-1}_{\max(a^q_n(y)), \max(a_n^{x,\nu}(y))}{}'' A_n^{x,\nu}(y)$.

This is possible since there is a maximal element for the a's unboundedly often. And by choosing the a^q_n's inductively for n, we maintain the last item of 2.4. Then, for $n > 1$, let $A^q_n = A'_n \cap \{x \mid \nu \in x \to \pi_{\beta^q_n, \beta^p_n}(\nu) \in x\}$.
For every $\langle x, \nu\rangle$ and $h \in [\prod_{i>1} A_i^{x,\nu} \times B_i^{x,\nu}]^{<\omega}$, let $\pi_{x,\nu,p}(h)$ be the corresponding pointwise projection of h from the maximal coordinates of $p_{x,\nu}$ to p. Let $\pi_{q,x,\nu}(h)$ be the projection from the maximal coordinates of q to $p_{x,\nu}$, and let $\pi_{q,p}(h)$ be the projection from the maximal coordinates of q to p.

Since every $p_{x,\nu} \leq p$, let $A_0^{x,\nu} \in U_0$ be such that for all $y \in A_0^{x,\nu}$, for all $h \in [\prod_{i>1} A_i^{x,\nu} \times B_i^{x,\nu}]^{<\omega}$ with $y \prec h$, $f_0^{x,\nu}(y)(h) \leq f^p_0(\langle x, \nu\rangle^\frown \pi_{x,\nu,p}(h))$. For all $y \in \mathcal{P}_\kappa(\kappa_0)$, and $x \in A^q_1$, $\nu \in B^q_1 = B^p_1$ with $\nu \in x$, set $f^q_0(y)(\langle x, \nu\rangle) = f^p_0(y)(\langle x, \nu\rangle)$, and

- if $y \in A_0^{x,\nu}$, set $f_0^q(y)(\langle x,\nu\rangle^\frown h) = f_0^{x,\nu}(\pi_{q,x,\nu}(h))$,
- otherwise, set $f_0^q(y)(\langle x,\nu\rangle^\frown h) = f_0^p(y)(\langle x,\nu\rangle^\frown \pi_{q,p}(h))$.

For all $y \in A_1^q$, for each h with $y \prec h$, set

$$f_1^q(y)(h) = \bigcup_{x,\nu:\nu\in x, x\prec y} f_1^{x,\nu}(y)(\pi_{q,x,\nu}(h)) \restriction \mathrm{dom}(f_1^{x,\nu}(y)(\pi_{q,x,\nu}(h))) \setminus a_1^p(y).$$

Then set $F_1^q(y) = \langle a_1^q(y), A_1^q(y), f_1^q(y)\rangle$.

For $n > 1$ and $y \in A_n^q$, set $f_n^q(y)(h) = \bigcup_{x\prec y,\nu\in x} f_n^{x,\nu}(y)(\pi_{q,p}(h))$ and $F_n^q(y) = \langle a_n^q(y), A_n^q(y), f_n^q(y)\rangle$. Then q is as desired.

□

Corollary 4.2. *Suppose that p is a condition, D is an open dense set, and $n > \mathrm{lh}(p)$. Then there is a condition $q \leq^* p$ such that for all $r \leq q$ with length n, if there is $r' \leq^* r$ in D, then r is in D.*

Proof. By induction on $n-l$. If $n = \mathrm{lh}(p)+1$, the result follows from the Diagonal lemma. Suppose $n > \mathrm{lh}(p)+1$. For every $\langle x,\nu\rangle$, such that $p^\frown\langle x,\nu\rangle$ is defined, by the inductive assumption let $p_{x,\nu} \leq^* p^\frown\langle x,\nu\rangle$ be such that for all $r \leq p_{x,\nu}$ with length n, if there is $r' \leq^* r$ in D, then r is in D.

Defining these condition inductively, we arrange that they satisfy the assumptions of the diagonal lemma. Apply the diagonal lemma to the conditions $p_{x,\nu}$ and p to get $q \leq^* p$, such that $q^\frown\langle x,\nu\rangle \leq^* p_{x,\nu}$, for all x,ν. Then q is as desired. For if $r \leq q$ is with length n, let x,ν be such that $r \leq p_{x,\nu}$. Now, if $r' \leq^* r$ is in D, then by the way we chose $p_{x,\nu}$, it follows that r is in D.

□

Remark 1. We can define q as above so that for all $l \leq k < n$ and $x \in A_k^q$, $a_k^p(x) = a_k^q(x)$. That is because when running the argument above, by induction, we may assume that for all $l < k < n$, for all x,ν and $y \in A_k^{p_{x,\nu}}$, $a_k^{p_{x,\nu}}(y) = a_k^p(y)$. Then, as in the proof of the Diagonal lemma, when diagonalizing over the $p_{x,\nu}$'s we get that for all $l \leq k < n$ and $x \in A_k^q$, $a_k^q(x) = a_k^p(x)$.

Lemma 4.3. *(Prikry lemma) Suppose that D is an open dense set and p is a condition with length l. Then there is some n and $q \leq^* p$, such that for all $\vec{x},\vec{\nu}$ of length n, such that $q^\frown\langle\vec{x},\vec{\nu}\rangle$ is defined, we have that $q^\frown\langle\vec{x},\vec{\nu}\rangle \in D$.*

Proof. First by shrinking measure one sets, we may assume that for some fixed n, for all $r \leq p$ of length $n+l$, there is some $r' \leq^* r$ such that $r' \in D$. Let $q \leq^* p$ be given by the above corollary applied to D. Then every n-step extension of q is in D.

□

Lemma 4.4. *For every $p \in \mathbb{P}$ and formula ϕ, there is $q \leq^* p$, such that q decides ϕ.*

Proof. Apply the Prikry lemma for the set $\{q \mid q \parallel \phi\}$ to find $p' \leq p$ and n, such that every n-step extension of p' is in D'. Then by shrinking measure one sets, in a rather standard way, we obtain $q \leq^* p'$, such that all n-step extensions of q decide ϕ the same way. Then q decides ϕ.

□

Corollary 4.5. $\mathbb{P}$ *does not add bounded subsets of κ*

Proof. This follows from the Prikry property and since $\langle \mathbb{P}, \leq^* \rangle$ is κ-closed. $\square$

Corollary 4.6. *$\mathbb{P}$ preserves cardinals up to and including κ.*

5. The generic extension

Prepare the ground model V, such that the supercompactness of κ is preserved by forcing with $\mathbb{P}_0$. Since $\mathbb{P}_0$ is κ_0-closed, and so does not add subsets of κ, by starting with a model of GCH, we have that in V, $2^\tau = \tau$ for all inaccessible $\tau < \kappa$. Also, in V, $\mathrm{GCH}_{\geq \kappa}$ holds.

Let G be $\mathbb{P}$-generic. Let $\langle x_n \mid n < \omega \rangle$ be the diagonal supercompact Prikry sequence added by G. Then $\bigcup_n x_n = \kappa_\omega$ and $V[G] \models (\forall i < \omega) \operatorname{cf}(\kappa_i) = \omega$ and $\mu = \kappa^+$. Next we show that the forcing blows up the powerset of κ.

Lemma 5.1. *Suppose $n < \omega, \alpha < \mu^+$, and p is such that $n \geq \mathrm{lh}(p)$ and for all $y \in A^p_n$, $\alpha \in a^p_n(y)$. Then $D_{n,\alpha} := \{q \mid \mathrm{lh}(q) > n, (\exists \beta := [x \mapsto \beta_x]_{U_n})(\forall_{U_n} x)(\forall h \in \mathrm{dom}(f^p_n(x))) f^p_n(x)(h)(\alpha) = \beta_x\}$ is dense below p.*

Proof. Let $q \leq p$ and $\mathrm{lh}(q) > n$. Say $q \leq^* p^\frown \langle \vec{x}, \vec{\nu} \rangle$, and let ν is the $n - \mathrm{lh}(p)$ - th element of the sequence $\vec{\nu}$. Then let $\beta := \pi_{[x \mapsto \max(a^p_n(x))]_{U_n}, j_n(\alpha)}(\nu)$. Denote $\beta = [x \mapsto \beta_x]_{U_n}$. Then by definition of the $\mathbb{Q}$-modules, we have that for U_n-almost all x, for all $h \in \mathrm{dom}(f^q_n(x))$, $f^q_n(x)(h)(\alpha) = \beta_x = \pi_{\max(a^p_n(x), \alpha)}(\nu_x)$.

$\square$

For p in $D_{n,\alpha}$, define $g^p_n(\alpha) = \beta$, where β witnesses that p is in that set. Let

$$F := \bigcup_{p \in G, n \geq \mathrm{lh}(p), y \in A^p_n} a^p_n(y).$$

Note that by genericity of the Prikry sequence and definition of $\mathbb{P}$, this is the same as taking $F = \bigcup_{p \in G, n \geq \mathrm{lh}(p)} a^p_n(x_n)$. Define $g^*_n : F \to \kappa_n$ by $g^*_n(\alpha) = g^p_n(\alpha)$ for some p in $G \cap D_{n,\alpha}$, if such exists, and 0 otherwise.

Lemma 5.2. *F is unbounded in μ^+*

Proof. Let $\alpha < \mu^+$. We claim that the set $D_\alpha := \{p \mid (\exists \alpha' > \alpha)(\exists i \geq \mathrm{lh}(p))(\forall y \in A^p_i)\alpha' \in a^p_i(y)\}$ is dense. Let p be given. Since:

$$\beta_0 := \sup_{n \geq \mathrm{lh}(p), y \in A^p_n, h \in \mathrm{dom}(f^p_n(y))} \mathrm{dom}(f^p_n(y)(h)) < \mu^+,$$

we have that $\beta := \max(\beta_0, \alpha) < \mu^+$. Take α' with $\beta < \alpha' < \mu^+$. Now we can extend p to a condition q, so that for some $n > \mathrm{lh}(q)$, for all $y \in A^q_n$, we have that $\alpha' \in a^q_n(y)$ $\square$

Remark 2. By a similar argument, we get that $F \cap \mu$ is unbounded in μ.

Lemma 5.3. *If $\alpha < \beta$ are both in F, then for all large n, $g^*_n(\alpha) < g^*_n(\beta)$.*

Proof. Let p_1, p_2 in G witness that $\alpha, \beta \in F$. We can find a common extension $p \in G$, such that for all $n \geq \mathrm{lh}(p)$, for all $y \in A^p_n$, $\{\alpha, \beta\} \subset a^p_n(y)$. We will show that for all $n \geq \mathrm{lh}(p)$, $g^*_n(\alpha) < g^*_n(\beta)$. To this end, let $q \in G$ be such that $q \leq p$ and $\mathrm{lh}(q) > n$. Let $q \leq^* p^\frown \langle \vec{x}, \vec{\nu} \rangle$, and let ν is the $n - \mathrm{lh}(p)$ - th element of the sequence $\vec{\nu}$. Then let $\delta := \pi_{[x \mapsto \max(a^p_n(x))]_{U_n}, j_n(\alpha)}(\nu)$ and $\delta' := \pi_{[x \mapsto \max(a^p_n(x))]_{U_n}, j_n(\beta)}(\nu)$. Then by definition of the $\mathbb{Q}$-modules, we have that for U_n-almost all x, for all $h \in \mathrm{dom}(f^q_n(x))$, $f^q_n(x)(h)(\alpha) = \delta_x < \delta'_x = f^q_n(x)(h)(\beta)$. So, $g^*_n(\alpha) = \delta < \delta' = g^*_n(\beta)$.

□

We have that every g_n^* has range κ_n. Next we use the genericity of $\langle x_n \mid n < \omega\rangle$ to define functions with ranges in $\kappa^n_{x_n} := |x_n|$. Now, for all η, let F_n^η be the function such that $[F_n^\eta]_{U_n} = \eta$. In $V[G]$, define functions $\langle t_\alpha \mid \alpha < \mu^+\rangle$ in $\prod_n \kappa^n_{x_n}$ by

$$t_\alpha(n) := F_n^{g_n^*(\alpha)}(x_n).$$

Then $\langle t_\alpha \mid \alpha \in F\rangle$ are increasing sequences in $\prod_n \kappa^n_{x_n}$ mod finite.

Corollary 5.4. *$V[G] \models 2^\kappa = \mu^+$.*

6. No very good scale

In this section we show that there is no very good scale at κ in $V[G]$. Suppose for contradiction, that in $V[G]$, $\langle f_\alpha \mid \alpha < \mu\rangle$ is a very good scale in some product $\prod_n \tau_n$, of regular cardinals with supremum κ. For every n there is some n', such that $\tau_n < \kappa_{x_{n'}}$. Suppose for simplicity that $n' = n$. The general case is similar. Also suppose for simplicity that all of this is forced by the empty condition.

Proposition 6.1. *For all $\alpha < \mu$ and $p \in \mathbb{P}_0$, there is $q \leq^* p$, such that every $n+1$-step extension of q decides a value of $\dot{f}_\alpha(n)$, and such that for all $k \leq n, x \in A^q_k$, $a^q_k(x) = a^p_k(x)$.*

Proof. Let $D := \{q \mid \exists\gamma(q \Vdash \dot{f}_\alpha(n) = \gamma)\}$; this is clearly a dense open set. Then by Corollary 4.2, we get $q \leq^* p$ such that for all $r \leq q$ with length $n+1$, if there is $r' \leq^* r$ in D, then r is in D.

Claim 6.2. *For all $r \leq p$ with $\mathrm{lh}(r) = n+1$, there is $r' \leq^* r$ with $r' \in D$.*

Proof. Fix such r; say $x := x^r_n$. Then $r \Vdash \dot{f}_\alpha(n) < \kappa_x$. Apply the Prikry property to "$\dot{f}_\alpha(n) = \gamma$", for all $\gamma < \kappa_x$, to construct a $\leq^*$-decreasing sequence $\langle r_\gamma \mid \gamma < \kappa_x\rangle$ of direct extensions of r, deciding these formulas. Then let r' be stronger than each r_γ; $r' \in D$. □

It follows that every $r \leq q$ with length $n+1$ is in D. Also, by Remark 1, for all $k \leq n, x \in A^q_k$, $a^q_k(x) = a^p_k(x)$. □

Remark 3. Since $(\mathbb{P}, \leq^*)$ is κ_0-closed, the above proposition also works for functions in $\prod_n \kappa^+_{x_n}$, $\prod_n \kappa^n_{x_n}$, $\prod_n (\kappa^n_{x_n})^+$, etc. (recall $\kappa^n_x = |x|$ for $x \in \mathcal{P}_\kappa(\kappa_n)$)

Now let H be $\mathbb{D}$-generic induced from G, and let G_0 be $\mathbb{P}_0/H$-generic over V. Since $\mathbb{P}/H$ has the μ-chain condition there is a club subset of μ, $E \in V[H]$, such that every point in E is very good, and of course E remains a club in $V[G_0]$.

For two functions f, g, we will write $f <_n g$ to denote that for all $k \geq n$, $f(k) < g(k)$.

Lemma 6.3. *In $V[G_0]$, there is $n < \omega$, and a $< \kappa$-club $C \subset \mu$, such that for all $\alpha < \beta$ in C, there is $p \in G_0$, such that $p \Vdash_{\mathbb{P}} \dot{f}_\alpha <_n \dot{f}_\beta$.*

Proof. For every $\delta < \mu$ with $\omega < \mathrm{cf}^V(\delta) = \mathrm{cf}^{V[G_0]}(\delta) < \kappa$, let $Y_\delta \in V$ be any club in δ of order type $\mathrm{cf}^V(\delta)$. Enumerate $\mathcal{P}^V(Y_\delta)$ by $\{C_{\delta,i} \mid i < 2^{\mathrm{cf}(\delta)}\}$. Since κ is strong limit, we have that $2^{\mathrm{cf}(\delta)} < \kappa$. So, by applying the Prikry property, we can produce a condition p_δ of length 0, such that for each i, and $n < \omega$, p_δ decides whether $C_{\delta,i}$ and n witness very goodness of δ. By density, we choose each $p_\delta \in G_0$. By assumption, for club many δ's there is some i, n such that $C_{\delta,i}$ and n witness very goodness.

Let $j : V[G_0] \to M$ be a μ-supercompact embedding with critical point κ. Set $\rho := \sup j"\mu$. Then by elementarity, there is a condition $p^* \in j(G_0)$, $n < \omega$, and $C^* \in M$ of order type $\mathrm{cf}^M(\rho) = \mu$, such that p^* forces that C^*, n witness that ρ is very good. Let $C := \{\gamma < \mu \mid j(\gamma) \in C^*\}$.

Then C is $< \kappa$ club in μ. Now suppose that $\alpha < \beta$ are in C and $q \in G_0$. Let $r^* \leq^* j(q), p^*$ be in $j(G_0)$. Then $r^* \Vdash_{j(\mathbb{P})} j(\dot{f})_{j(\alpha)} <_n j(\dot{f})_{j(\beta)}$ (since p^* forces it). So, by elementarity, there is a condition $p \in G_0$, $p \leq^* q$, such that $p \Vdash_{\mathbb{P}} \dot{f}_\alpha <_n \dot{f}_\beta$.

□

Let $\dot{C}$ be a $\mathbb{P}_0$ name for a club as above and suppose that the empty condition forces (over $\mathbb{P}_0$) that $\dot{C}, n$ are as above. I.e. for all $p \in \mathbb{P}_0$, and $\alpha < \beta < \mu$, if $p \Vdash_{\mathbb{P}_0} \alpha, \beta \in \dot{C}$, then there is $q \leq^* p$, such that $q \Vdash_{\mathbb{P}} \dot{f}_\alpha <_n \dot{f}_\beta$.

Lemma 6.4. *For all $\tau < \kappa_\omega$ and $p \in \mathbb{P}_0$, there is $X \subset \mu$ in V with $|X| = \tau$ and $r \leq^* p$, such that $r \Vdash_{\mathbb{P}_0} X \subset \dot{C}$.*

Proof. Let m be such that $\tau < \kappa_m$. We use the following claim.

Claim 6.5. *For all $\alpha < \mu$, for all p, there is $\beta > \alpha$ and $q \leq^* p$, such that $\pi_m(q) = \pi_m(p)$ and $q \Vdash_{\mathbb{P}_0} \beta \in \dot{C}$.*

Proof. Construct $\leq^*$-decreasing sequence of conditions $\langle q_k \mid k < \omega\rangle$ and an increasing sequence of points $\langle \alpha_k \mid k < \omega\rangle$, such that $\alpha_0 = \alpha$, every $q_k \Vdash_{\mathbb{P}_0} \dot{C} \cap (\alpha_k, \alpha_{k+1}] \neq \emptyset$, and $\pi_m(q_k) = \pi_m(p)$. We can do this by standard arguments since $\mathbb{P}_{0m}$ has the κ_{m-1}^{++}-c.c. and $\mathbb{P} \restriction [m, \omega)$ is κ_m-closed. Then let $\beta = \sup_k \alpha_k$ and let $q \leq^* q_k$ for all k. Then $q \Vdash_{\mathbb{P}_0} \beta \in \dot{C}$. □

Fix p. We will construct a sequence $\langle \beta_\eta \mid \eta < \tau\rangle$ and $\langle q_\eta \mid \eta < \tau\rangle$, such that for each η, $\pi_m(q_\eta) = \pi_m(p)$ and $\langle q_\eta \restriction [m, \omega) \mid \eta < \tau\rangle$ is $\leq^\sim$-decreasing.

Suppose we have defined the sequences up to η. Let $q \leq^* p$ be such that $\pi_m(q) = \pi_m(p)$ and $q \restriction [m, \omega) \leq^\sim q_\xi \restriction [m, \omega)$ for all $\xi < \eta$. Let $q_\eta \leq^* q, \beta_\eta > \sup_{\xi<\eta} \beta_\xi$ be given by the claim applied to q and $\sup_\xi \beta_\xi$.

Finally let $r \leq^* p$ be such that $\pi_m(r) = \pi_m(p)$ and $r \restriction [m, \omega) \leq^\sim q_\eta \restriction [m, \omega)$ for all $\eta < \tau$. Set $X = \{\beta_\eta \mid \eta < \tau\}$. Then $r \Vdash_{\mathbb{P}_0} X \subset \dot{C}$. □

Apply the above lemma to find a condition $r \in \mathbb{P}_0$ and $X \subset \mu$ of size κ_n^{++}, such that $r \Vdash_{\mathbb{P}_0} X \subset \dot{C}$. For every $\alpha \in X$, let $p_\alpha \leq^* r$ be given by Proposition 6.1. I.e. every $q \leq p_\alpha$ with length $n + 1$ decides $\dot{f}_\alpha(n)$, and for all $k \leq n, x \in A_k^q$, $a_k^{p_\alpha}(x) = a_k^r(x)$. $\mathbb{P} \restriction [n + 1, \omega)$ is κ_{n+1}-closed and $|X| = \kappa_n^{++}$. So by defining the p_α's inductively, we arrange that $\langle p_\alpha \restriction [n + 1, \omega) \mid \alpha \in X\rangle$ is $\leq^\sim$-decreasing.

Consider $\{\pi_{n+1}(p_\alpha) \mid \alpha \in X\} \subset \mathbb{P}_{0n+1}$. By the same Δ-system argument as in Proposition 3.3, there is an unbounded $X' \subset X$, such that $\{\pi_{n+1}(p_\alpha) \mid \alpha \in X'\}$ are pairwise compatible. But that means $\{p_\alpha \mid \alpha \in X'\}$ are pairwise compatible with respect to $\leq^*$. For all α, β in X', let $p_{\alpha,\beta} \leq^* p_\alpha, p_\beta$ be such that $p_{\alpha,\beta} \Vdash_{\mathbb{P}} \dot{f}_\alpha <_n \dot{f}_\beta$. Let $r_{\alpha,\beta} \leq p_{\alpha,\beta}$ be of length $n + 1$ and of the form $r_{\alpha,\beta} = p_{\alpha,\beta}^\frown\langle \vec{x}, \vec{\nu}\rangle$, for some $\vec{x}, \vec{\nu}$. But then since for all $k \leq n, x \in A_k^q$, $a_k^{p_\alpha}(x) = a_k^r(x)$, we have that there are $\vec{x}_{\alpha,\beta}, \vec{\nu}_{\alpha,\beta}$, such that:

- $r_{\alpha,\beta} \leq^* p_\alpha^\frown\langle \vec{x}_{\alpha,\beta}, \vec{\nu}_{\alpha,\beta}\rangle$
- $r_{\alpha,\beta} \leq^* p_\beta^\frown\langle \vec{x}_{\alpha,\beta}, \vec{\nu}_{\alpha,\beta}\rangle$

Denote $h_{\alpha,\beta} := \langle \vec{x}_{\alpha,\beta}, \vec{\nu}_{\alpha,\beta}\rangle$. The number of possible $h_{\alpha,\beta}$'s is κ_n, and $|X'| = \kappa_n^{++} = (2^{\kappa_n})^+$. By Erdos-Rado, the function $\langle \alpha, \beta\rangle \mapsto h_{\alpha,\beta}$ has a homogenous set Y is size κ_n^+. Let $\langle \vec{x}, \vec{\nu}\rangle = h_{\alpha,\beta}$ for all α, β in Y.

For all $\alpha \in Y$, let $\gamma_\alpha < \kappa$ be such that, $p_\alpha ^\frown \langle \vec{x}, \vec{\nu}\rangle \Vdash \dot{f}_\alpha(n) = \gamma_\alpha$. (Here we use that p_α is as in the conclusion of Proposition 6.1.) Suppose that $\alpha < \beta$ are both in Y. Since $r_{\alpha,\beta} \leq p_{\alpha,\beta}$ and $p_{\alpha,\beta} \Vdash \dot{f}_\alpha <_n \dot{f}_\beta$, we have that $r_{\alpha,\beta} \Vdash \dot{f}_\alpha(n) < \dot{f}_\beta(n)$. But $r_{\alpha,\beta} \leq^* p_\alpha ^\frown \langle \vec{x}, \vec{\nu}\rangle, p_\beta ^\frown \langle \vec{x}, \vec{\nu}\rangle$, so $\gamma_\alpha < \gamma_\beta$.

But then $\{\gamma_\alpha \mid \alpha \in Y\}$ is a subset of κ of size κ_n^+. Contradiction.

7. Bad scale

Recall that we prepared the ground model V, so that the supercompactness of κ is preserved by forcing with $\mathbb{P}_0$. In V, fix a scale $\langle g^*_\alpha \mid \gamma < \mu\rangle \in V$ in $\prod_n \kappa_n^+$. Set $S := \{\gamma < \mu \mid \omega < \mathrm{cf}(\gamma) < \kappa, \gamma$ is a bad point for $\langle g^*_\alpha \mid \gamma < \mu\rangle\}$. By standard reflection arguments S is stationary in V. Also, since $\mathbb{P}_0$ preserves μ and is κ^+-closed, $\langle g^*_\alpha \mid \gamma < \mu\rangle$ remains a bad scale after forcing with $\mathbb{P}_0$. More precisely, if G_0 is $\mathbb{P}_0$-generic, a point of cofinality less than κ is bad in V iff it is bad in $V[G_0]$, and the set S is stationary in $V[G_0]$ (since κ remains supercompact in $V[G_0]$).

So if H is $\mathbb{D}$-generic, since $\mathbb{P}_0$ projects to $\mathbb{D}$, we have that S is stationary in $V[H]$. Then by the μ-chain condition of $\mathbb{P}/\mathbb{D}$, S is stationary after forcing with $\mathbb{P}$.

The next lemma will be used to show that a witness of goodness in the generic extension gives rise to a witness of goodness in the ground model. In particular, if a point is bad in V, then it is bad in $V[G]$.

Lemma 7.1. *Let $\tau < \kappa$ be a regular uncountable cardinal in V (and so in $V[G]$), and suppose $V[G] \models A \subset ON, o.t.(A) = \tau$. Then there is a $B \in V$ such that B is an unbounded subset of A.*

Proof. Let $p \in G$, $p \Vdash \dot{h} : \tau \to \dot{A}$ enumerate $\dot{A}$. By the Prikry lemma, define a $\leq^*$-decreasing sequence $\langle p_\alpha \mid \alpha < \tau\rangle$, such for every $\alpha < \tau$, $p_\alpha \leq^* p$ and there is $n_\alpha < \omega$, such that every $q \leq p_\alpha$ with length n_α decides $\dot{h}(\alpha)$. Then there is an unbounded $I \subset \tau$ and $n < \omega$ such that for all $\alpha \in I$, $n = n_\alpha$. Let p' be stronger than all p_α for $\alpha < \tau$. By appealing to density, we may assume that $p' \in G$. Let $q \leq p$ be a condition in G with length n, and set $B = \{\gamma \mid (\exists \alpha \in I) q \Vdash \dot{h}(\alpha) = \gamma\}$. Then B is as desired.

□

Note that the above lemma already implies that the approachability property fails in $V[G]$, and so weak square also fails.

Recall that for every $x \in \mathcal{P}_\kappa(\kappa_n)$, κ^n_x denotes $|x|$, which is a cardinal on a U_n-measure one set. Also, $\forall n < \omega$, $\forall \eta < \kappa_n^+$, we fixed $F^\eta_n : \mathcal{P}_\kappa(\kappa_n) \longrightarrow V$, such that $[F^\eta_n]_{U_n} = \eta$. We may assume that $\forall x F^\eta_n(x) < (\kappa^n_x)^+$. Define in $V[G]$, $\langle g_\beta \mid \beta < \mu\rangle$ in $\prod_n (\kappa^n_{x_n})^+$ by:

$$g_\beta(n) = F_n^{g^*_\beta(n)}(x_n)$$

To show that this is a scale we need the following bounding lemma.

Lemma 7.2. *Suppose that in $V[G]$, $h \in \prod_n (\kappa^n_{x_n})^+$. Then there is a sequence of functions $\langle H_n \mid n < \omega\rangle$ in V, such that $\mathrm{dom}(H_n) = \mathcal{P}_\kappa(\kappa_n)$, $H_n(x) < (\kappa^n_x)^+$ for all x, and for all large n, $h(n) \leq H_n(x_n)$.*

Proof. Let p force that $\dot{h} \in \prod_n (\kappa^n_{\dot{x}_n})^+$. For simplicity, say $\mathrm{lh}(p) = 0$.

Fix $n < \omega$. Let $p_n \leq^* p$ be such that every $n+1$-step extension decides $\dot{h}(n)$. Let $q \leq^* p_n$, for all n. For all $\vec{z}, \vec{\nu}$ of length $n+1$, such that $q^\frown\langle \vec{z}, \vec{\nu}\rangle$ is defined, let $\gamma_{\vec{z},\vec{\nu}}$ be such that $q^\frown\langle \vec{z}, \vec{\nu}\rangle \Vdash \dot{h}(n) = \gamma_{\vec{z},\vec{\nu}}$. For $x \in A^q_n, \nu \in B^q_n$ with $\nu \in x$, define $H_n(x,\nu) = \sup\{\gamma_{\vec{z},\vec{\nu}} \mid z_n = x, \nu_n = \nu\} < \kappa^n_x$, where z_n and ν_n denote the last elements of $\vec{z}$ and $\vec{\nu}$ respectively. Let $H_n(x) = \sup_{\nu \in B^q_n, \nu \in x} H_n(x,\nu) < (\kappa^n_x)^+$.

Then q forces that $\langle H_n \mid n < \omega\rangle$ is as desired.

□

Corollary 7.3. *$\langle g_\beta \mid \beta < \mu\rangle$ is a bad scale in $V[G]$*

Proof. $\langle g_\beta \mid \beta < \mu\rangle$ is a scale by the way we defined it and Lemma 7.2, (see for example the arguments in [1]). Also, by Lemma 7.1, if γ is a good point in $V[G]$ for $\langle g_\beta \mid \beta < \mu\rangle$ with cofinality τ with $\omega < \tau < \kappa$, then γ is a good point in V for $\langle g^*_\beta \mid \beta < \mu\rangle$. Finally, the set of bad points S is still stationary in $V[G]$.

□

We conclude with some questions.

Question 1. *What can be said about the tree property at κ in the above construction?*

Question 2. *Can we use short extenders and collapses to obtain the present construction for $\kappa = \aleph_\omega$?*

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